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A NOTE ON QUADRATIC JORDAN ALGEBRAS OF DEGREE 3(1)

BY
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Abstract. McCrimmon has defined a class of quadratic Jordan algebras of degree 3 obtained from a cubic form, a quadratic mapping and a base point. The structure of such an algebra containing no absolute zero-divisor is determined directly. A simple proof of Springer's Theorem on isomorphism of reduced simple exceptional quadratic Jordan algebras is given.

1. **Introduction and basic concepts.** A *unital quadratic Jordan algebra* over a field Φ is a triple $(\mathscr{J}, U, 1)$ where $\mathscr{J}$ is a Φ vector space, 1 a distinguished element of $\mathscr{J}$, and U is a mapping $a \to U_a$ of $\mathscr{J}$ into $\mathrm{End}_\Phi(\mathscr{J})$ satisfying the following axioms:

QJ1. U is Φ-quadratic, that is $U_{\lambda a} = \lambda^2 U_a$, $\lambda \in \Phi$, $a \in \mathscr{J}$, and $U_{a,b} = U_{a+b} - U_a - U_b$ is Φ-bilinear in a and b.

QJ2. $U_1 = 1$.

QJ3. $U_{aU_b} = U_b U_a U_b$.

QJ4. If $V_{a,b}$ is defined by $xV_{a,b} = aU_{x,b}$, then $U_b V_{a,b} = V_{b,a} U_b$.

QJ5. QJ(1)–(4) hold for $\mathscr{J}^P = \mathscr{J} \otimes_\Phi P$, P any field extension of Φ.

Powers are defined inductively: $x^0 = 1$, $x^1 = x$, and $x^{n+2} = x^n U_x$. An element $z \in \mathscr{J}$, $z \neq 0$, is said to be an *absolute zero divisor* if $U_z = 0$. An element $a \in \mathscr{J}$ is *invertible* if there exists a $b \in \mathscr{J}$ with $bU_a = a$ and $b^2 U_a = 1$; such a b is uniquely determined and is denoted a^{-1}. We say that $\mathscr{J}$ is a *quadratic Jordan division algebra* if every nonzero element of $\mathscr{J}$ is invertible.

Let $\mathscr{J}$ be a Φ vector space. Assume given a quadratic form Q on $\mathscr{J}$ and a distinguished element $1 \in \mathscr{J}$ with $Q(1) = 1$. Define $aU_b = Q(a, b^*)a - Q(a)b^*$ where $b^* = Q(b, 1)1 - b$ and $Q(a, b) = Q(a+b) - Q(a) - Q(b)$. Jacobson and McCrimmon have shown in [5] that this defines a quadratic Jordan algebra, the quadratic Jordan algebra of the quadratic form Q with base point 1, denoted $\mathscr{J}(Q, 1)$.

Let $\mathscr{J}$ be a Φ vector space. Assume given (i) a cubic form N on $\mathscr{J}$ with values in Φ (so N is homogeneous of degree 3 and $N(x+y) = N(x) + \Delta_x^y N + \Delta_y^x N + N(y)$, where $\Delta_x^y N$ is the directional derivative of N in the direction y, evaluated at x),

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(ii) a quadratic mapping $x \to x^{\#}$ in $\mathscr{J}$, and (iii) a distinguished element $1 \in \mathscr{J}$, related by

(1) $x^{\#\#} = N(x)x$.

(2) $N(1) = 1$.

(3) $T(x^{\#}, y) = \Delta_x^y N$, where $T(x, y) = -\Delta_1^x \Delta^y \log N$.

(4) $1^{\#} = 1$.

(5) $1 \times y = T(y)1 - y$, where $T(y) = T(y, 1)$ and $x \times y = (x+y)^{\#} - x^{\#} - y^{\#}$.

Assume moreover that these hold for any $\mathscr{J}^{\mathrm{P}}$, P any field extension of Φ. Introduce a U operator.

(6) $yU_x = T(x, y)x - x^{\#} \times y$.

McCrimmon has shown in [6] that $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra which we denote $\mathscr{J}(N, \#, 1)$.

We recall also that a *composition algebra* $\mathscr{C}$ over Φ is a unital Φ-algebra (not necessarily associative; unital = contains a unit element 1) with a nondegenerate quadratic form n of $\mathscr{C}$ into Φ such that $n(1) = 1$ and $n(ab) = n(a)n(b)$. We refer to [1] for the determination of composition algebras and their properties. Let $\mathscr{C}$ be a composition algebra over Φ, $\mathscr{C}_3$ the algebra of 3×3 matrices over $\mathscr{C}$, γ the diagonal matrix diag $\{\gamma_1, \gamma_2, \gamma_3\}$ where the $\gamma_i \neq 0$ are in Φ. Then $J_\gamma \colon x \to \gamma^{-1}\bar{x}^t\gamma$ is an involution in $\mathscr{C}_3$ if $\bar{x} = (\bar{x}_{ij})$ for $x = (x_{ij})$ and x^t is the transpose of x. Let $\mathscr{H}(\mathscr{C}_3, J_\gamma)$ be the Φ-space of matrices satisfying $x^{J_\gamma} = x$ and whose diagonal entries lie in Φ. McCrimmon has shown in [6] that if one defines

$$N(x) = \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_3^{-1}\gamma_2 n(a_1) - \alpha_2\gamma_1^{-1}\gamma_3 n(a_2) - \alpha_3\gamma_2^{-1}\gamma_1 n(a_3) - t(a_1a_2a_3), \tag{7}$$

$$x^{\#} = \begin{pmatrix} \alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2 n(a_1) & \gamma_1^{-1}\gamma_2\overline{a_1a_2} - \alpha_3 a_3 & a_3a_1 - \gamma_1^{-1}\gamma_3\alpha_2\bar{a}_2 \\ a_1a_2 - \gamma_2^{-1}\gamma_1\alpha_3\bar{a}_3 & \alpha_3\alpha_1 - \gamma_1^{-1}\gamma_3 n(a_2) & \gamma_2^{-1}\gamma_3\overline{a_2a_3} - \alpha_1 a_1 \\ \gamma_3^{-1}\gamma_1\overline{a_3a_1} - \alpha_2 a_2 & a_2a_3 - \gamma_3^{-1}\gamma_2\alpha_1\bar{a}_1 & \alpha_1\alpha_2 - \gamma_2^{-1}\gamma_1 n(a_3) \end{pmatrix} \tag{8}$$

for

$$x = \begin{pmatrix} \alpha_1 & a_3 & \gamma_1^{-1}\gamma_3\bar{a}_2 \\ \gamma_2^{-1}\gamma_1\bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \gamma_3^{-1}\gamma_2\bar{a}_1 & \alpha_3 \end{pmatrix},$$

then the cubic form N on $\mathscr{J} = \mathscr{H}(\mathscr{C}_3, J_\gamma)$, 1 the unit matrix and $x^{\#}$ satisfy (1)–(5) in $\mathscr{J}^{\mathrm{P}}$ for every P. Hence if U is defined by (6) then $\mathscr{J}(N, \#, 1)$ is a quadratic Jordan algebra. From now on $\mathscr{H}(\mathscr{C}_3, J_\gamma)$ will denote the space $\mathscr{H}(\mathscr{C}_3, J_\gamma)$ with this quadratic Jordan structure.

The following theorem has been proved by Springer [8] when the characteristic of $\Phi \neq 2$ or 3.

THEOREM 1. *Let $\mathscr{J} = \mathscr{J}(N, \#, 1)$ be a unital quadratic Jordan algebra over a field Φ. If $\mathscr{J}$ has no absolute zero divisor then $\mathscr{J}$ is either* (i) *a quadratic Jordan division algebra,* (ii) *a direct sum $\Phi \oplus \mathscr{J}(Q, 1')$, where $\mathscr{J}(Q, 1')$ is the quadratic Jordan algebra of the quadratic form Q with base point $1'$, or* (iii) $\mathscr{H}(\mathscr{C}_3, J_\gamma)$.

This result follows also from structure theory [4]. Our purpose is to give a direct proof of this theorem which depends only on the properties of N and $\#$ obtained in [6] and on those of $\mathcal{J}(Q, 1)$ found in [5]. We also prove Springer's Theorem on isomorphism of $\mathcal{H}(\mathcal{C}_3, J_\gamma)$'s. This result is new in characteristic 2.

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2. **The proof of Theorem 1.** Let $\mathcal{J}$ be any Φ vector space equipped with a cubic norm N, a mapping $\#$ and an element 1 satisfying (1)–(5), so we have the quadratic Jordan algebra $\mathcal{J}(N, \#, 1)$. The following identities can be found in [6]. Let $x, y, z \in \mathcal{J}$.

(9) $T(1)=3$.

(10) $x^\# \times y^\# + (x\times y)^\# = T(x^\#, y)y + T(y^\#, x)x$.

(11) $x^\# \times (y\times z) + (x\times y)\times(x\times z) = T(x^\#, y)z + T(x^\#, z)y + T(y\times z, x)x$.

(12) $T(x\times y, z) = T(x, y\times z)$.

(13) $T(x\times y) = T(x)T(y) - T(x, y)$.

(14) $x^\# \times x = [T(x^\#)T(x) - N(x)]1 - T(x^\#)x - T(x)x^\#$.

(15) $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$, where

(16) $S(x) = T(x^\#)$.

(17) $x^\# = x^2 - T(x)x + S(x)1$.

(18) $(yU_x)^\# = y^\# U_{x^\#}$.

Also an element $x \in \mathcal{J}$ is invertible if and only if $N(x)\neq 0$, in which case

(19) $x^{-1} = N(x)^{-1}x^\#$.

(20) $T(xU_y, z) = T(x, zU_y)$ follows immediately from (12) and (6).

From (5) $x = -x\times 1 + T(x)1$, so $x\times(x\times y) = -(x\times 1)\times(x\times y) + T(x)1\times(x\times y)$ $= x^\# \times (1\times y) - T(x^\#, y)1 - T(x^\#, 1)y - T(1\times y, x)x + T(x)[T(x\times y)1 - x\times y]$ $= x^\# \times (T(y)1 - y) - T(x^\#, y)1 - T(x^\#)y - T(1, x\times y)x + T(x)T(x\times y)1 - T(x)x\times y$ $= T(y)T(x^\#)1 - T(y)x^\# - x^\# \times y - T(x^\#, y)1 - T(x^\#)y - T(x\times y)x + T(x)T(x\times y)1$ $- T(x)x\times y = [T(x^\# \times y) + T(x)T(x\times y)]1 - T(y)x^\# - x^\#\times y - T(x^\#)y - T(x\times y)x$ $- T(x)x\times y$ by (11), (12), (13) and (5) and we have

$$(21)\quad \begin{aligned} x\times(x\times y) = {} & [T(x^\#\times y) + T(x)T(x\times y)]1 \\ & - T(y)x^\# - x^\#\times y - T(x^\#)y - T(x\times y)x - T(x)x\times y. \end{aligned}$$

An idempotent $e\neq 0$ is said to be *primitive* if $e^\# = 0$. (This implies $T(e)=1$, and the converse holds if the characteristic of $\Phi\neq 2$.)

LEMMA 1. *If $\mathcal{J} = \mathcal{J}(N, \#, 1)$ does not contain any absolute zero divisors and is not a quadratic Jordan division algebra then $\mathcal{J}$ contains a primitive idempotent.*

Proof. If $\mathcal{J}$ is not a quadratic Jordan division algebra, $N(x)=0$ for some $x\neq 0$, so $x^{\#\#}=0$ and there is a $y\neq 0$ (either $y=x^\#$ if $x^\#\neq 0$, or $y=x$ if $x^\#=0$) with $y^\#=0$. Then $y^2 = T(y)y$; if $T(y)\neq 0$ then $e = T(y)^{-1}y$ is an idempotent and $e^\# = 0$, while if

$T(y)=0$, $y^2=0$. Suppose $T(y)=0$. Since y is not an absolute zero divisor and $zU_y=T(y,z)y-y^\# \times z=T(y,z)y$, there is a z such that $T(y,z)=1$. Let $a=y\times z$, $T(a)=T(y\times z)=T(y)T(z)-T(y,z)=-1$ and $a\neq 0$; $a^\#=(y\times z)^\#=T(y^\#,z)z+T(z^\#,y)y-y^\#\times z^\#=T(z^\#,y)y$, hence $T(a^\#)=0=N(a)$. Let $e=a^\#-a$; $T(e)=1$ and $e^\# =(a^\#-a)^\#=a^{\#\#}-a^\#\times a+a^\#=N(a)a-[T(a^\#)T(a)-N(a)]1+T(a^\#)a+T(a)a^\#+a^\# =-a^\#+a^\#=0$. Hence $0=e^\#=e^2-T(e)e+T(e^\#)1=e^2-e$ and e is a primitive idempotent.

Assume from now on that $\mathscr{J}$ does not contain absolute zero divisors and is not a quadratic Jordan division algebra. Therefore $\mathscr{J}$ contains a primitive idempotent e. Let $f=1-e$; $1-e=T(e)1-e=1\times e=(e+f)\times e=e\times e+f\times e=2e^\#+f\times e=f\times e$ and $1=1^\#=(e+f)^\#=e^\#+e\times f+f^\#=e\times f+f^\#$, so

(22) $e\times f=f, f^\#=e$.

By (12) $T(f,e)=T(f\times e,e)=T(f,e\times e)=T(f,2e^\#)=0$, so we have

(23) $T(e,f)=0$.

Note that $T(f)=T(1-e)=T(1)-T(e)=2$ and $e=f^\#=f^2-T(f)f+T(f^\#)1=f^2-2f+1=f^2-f+e$, so $f^2=f$ and f is an idempotent. Therefore $(U_f)^2=U_fU_1U_f=U_{1U_f}=U_{f^2}=U_f$; similarly $(U_e)^2=U_e$. Moreover $xU_eU_f=T(e,x)eU_f=T(e,x)[T(f,e)f-f^\#\times e]=-T(e,x)e\times e=0$ and $xU_fU_e=T(f,x)fU_e-(e\times x)U_e=T(f,x)T(e,f)e-T(e\times x,e)e=-T(x,e\times e)e=0$, so $U_eU_f=0=U_fU_e$. Therefore U_e, U_f and $U_{e,f}=U_1-U_e-U_f$ are orthogonal projections; hence

(24) $\mathscr{J}=\mathscr{J}_1\oplus\mathscr{J}_{1/2}\oplus\mathscr{J}_0$ where $\mathscr{J}_1=\mathscr{J}U_e$, $\mathscr{J}_{1/2}=\mathscr{J}U_{e,f}$ and $\mathscr{J}_0=\mathscr{J}U_f$.

This is the Peirce decomposition of $\mathscr{J}$ with respect to e and $\mathscr{J}_i$ will be denoted $\mathscr{J}_i(e)$, if needed, to emphasize that the decomposition is taken with respect to e. By (20) we have

(25) $T(\mathscr{J}_i,\mathscr{J}_j)=0$ if $i\neq j$, $i,j=0,\frac{1}{2},1$.

In particular $e=1U_e\in\mathscr{J}_1$ and $f=1U_f\in\mathscr{J}_0$, so

(26) $T(\mathscr{J}_{1/2},e)=0$, $T(\mathscr{J}_{1/2},f)=0$ and $T(\mathscr{J}_{1/2})=T(\mathscr{J}_{1/2},1)=0$.

Note that

$$\begin{aligned} xU_e &= T(x,e)e,\\ xU_f &= T(x,f)f-e\times x, \qquad (27)\\ xU_{e,f} &= T(x,e)f+T(x,f)e-f\times x. \end{aligned}$$

Let $x\in\mathscr{J}_{1/2}$, then (26) and (27) imply $x=-f\times x$, and $1\times x=T(x)1-x=-x$ implies $e\times x=0$. Conversely if $x\in\mathscr{J}$ with $T(x)=0$ and $e\times x=0$ then $x=-f\times x$ and $0=T(x)=-T(f\times x)=T(f,x)-T(x)T(f)=T(f,x)$ so $T(e,x)=0$ and by (27) $x\in\mathscr{J}_{1/2}$.

(28) $x\in\mathscr{J}_{1/2}(e)$ if and only if $T(x)=0$ and $e\times x=0$.

If $y\in\mathscr{J}$ with $T(y)=0$ and $e\times y=-y$ then $0=T(y)=-T(e\times y)=T(e,y)-T(e)T(y)=T(e,y)$ so $T(f,y)=0$ and by (27) $y\in\mathscr{J}_0(e)$.

(29) $T(y)=0$ and $e\times y=-y$ imply $y\in\mathscr{J}_0(e)$.

Let $x\in\mathscr{J}_{1/2}$, then $x=-f\times x$ and $x^\#=(-f\times x)^\#=T(f^\#,x)x+T(x^\#,f)f-f^\#\times x^\# =T(x^\#,f)f-e\times x^\#\in\mathscr{J}_0$ by (10), (22), (26) and (27), so

(30) $x^\#\in\mathscr{J}_0$ for $x\in\mathscr{J}_{1/2}$.

If $x \in \mathscr{J}_0$ then $x^\# = (xU_f)^\# = x^\# U_{f^\#} = x^\# U_e = T(x^\#, e)e$ and $S(x) = T(x^\#) = T(x^\#, e)T(e) = T(x^\#, e)$, so we have

(31) $x^\# = S(x)e \in \mathscr{J}_1$ for $x \in \mathscr{J}_0$.

LEMMA 2 (FAULKNER [2]). *$\mathscr{J}_0 = \mathscr{J}(S, f)$, the quadratic Jordan algebra of the quadratic form S restricted to $\mathscr{J}_0$ with base point f.*

Proof. Let $x, y \in \mathscr{J}_0$, then $x = T(x, f)f - e \times x$. Define $x^* = e \times x = T(x, f)f - x$. We have $yU_x = T(x, y)x - x^\# \times y = T(x, y)x - S(x)e \times y = T(x, y)x - S(x)y^*$. But $S(x, y^*) = T(x \times y^*) = T(x)T(y^*) - T(x, y^*) = T(x)T(e \times y) - T(x, e \times y) = T(x)T(y, e \times 1) - T(x \times y, e) = T(x)T(y, f) - T(x \times y) = T(x)T(y) - T(x \times y) = T(x, y)$ by (12), (13), (25) and (31). Hence $yU_x = S(x, y^*)x - S(x)y^*$. Finally $S(f) = T(f^\#) = T(e) = 1$ and $f^* = e \times f = f$, so that $x^* = T(x, f)f - x = S(x, f^*)f - x = S(x, f)f - x$.

We shall now show that if $\mathscr{J}_{1/2} = \{0\}$ then we have case (ii), that is $\mathscr{J} = \mathscr{J}_1 \oplus \mathscr{J}_0$ is an algebra direct sum. Let $x, y \in \mathscr{J}_0$; $xU_e = T(e, x)e = 0$ by (26), $eU_x = T(x, e)x - x^\# \times e = -S(x)e \times e = 0$, $eU_{e,x} = T(e, e)x + T(e, x)e - (e \times x) \times e = x - e \times (T(x)f - x) = x - T(x)f + e \times x = 0$ by (27) and (22). Finally $yU_{e,x} = T(y, e)x + T(y, x)e - (e \times x) \times y = T(y, x)e - (T(x)f - x) \times y = T(x, y)e - T(x)(1 \times y - e \times y) - T(x \times y)e = T(x, y)e - T(x)(T(y)1 - y - T(y)f + y) - T(x \times y)e = 0$ by (13), (27) and (31). Hence if $x, x' \in \mathscr{J}_1$ and $y, y' \in \mathscr{J}_0$ then $(x+y)U_{x'+y'} = xU_{x'} + yU_{y'}$ so we have the algebra direct decomposition $\mathscr{J} = \mathscr{J}_0 \oplus \mathscr{J}_1$.

Assume we are not in case (ii), so $\mathscr{J}_{1/2} \neq \{0\}$. Suppose $(\mathscr{J}_{1/2})^\# = \{0\}$. Since $x \in \mathscr{J}_{1/2}$, $x \neq 0$, is not an absolute zero divisor, there exists a y, which by (25) may be taken in $\mathscr{J}_{1/2}$, with $T(x, y) \neq 0$. But, by (26), $T(x) = 0$ and $x \times y = (x+y)^\# - x^\# - y^\# = 0$ which yields $T(x, y) = T(x)T(y) - T(x \times y) = 0$, a contradiction. Therefore $\mathscr{J}_{1/2} \neq \{0\}$ implies $(\mathscr{J}_{1/2})^\# \neq \{0\}$. Let $x \in \mathscr{J}_{1/2}$ with $x^\# \neq 0$, then $x^\# \in \mathscr{J}_0$ by (30) and $x^{\#\#} \in \mathscr{J}_1$, by (31). However $x^{\#\#} = N(x)x \in \mathscr{J}_{1/2}$. Therefore $x^{\#\#} = 0$ and $N(x) = 0$. Since $e^\# = 0$, (27) and (31) imply

$$(32) \qquad N(x) = 0 \quad \text{for } x \in \mathscr{J}_i, \quad i = 0, \tfrac{1}{2}, 1.$$

Also if $x \in \mathscr{J}_{1/2}$ then $N(x) = 0$ implies that $S(x^\#) = T(x^{\#\#}) = T(N(x)x) = 0$. Since we can choose $x \in \mathscr{J}_{1/2}$ so that $x^\# \neq 0$ and $x^\# \in \mathscr{J}_0$ it is clear that the restriction of S to $\mathscr{J}_0$ is isotropic. Let $y \in \mathscr{J}_0$, $y \neq 0$. If $S(y) = 0$ then $y^\# = 0$ by (31) and y not an absolute zero divisor implies $T(y, z) \neq 0$ for some $z \in \mathscr{J}_0$. In the proof of Lemma 2 $T(y, z)$ was seen to equal $S(y, z^*)$, therefore the restriction of S to $\mathscr{J}_0$ is nondegenerate. From Theorem 11 of [5] we know that $\mathscr{J}_0$ contains a primitive idempotent unless characteristic $\Phi = 2$ and $T(\mathscr{J}_0) = 0$. In that case consider $e' = e + x + x^\#$, $x \in \mathscr{J}_{1/2}$, $x^\# \neq 0$; $T(e') = T(e) = 1$. We have $(e')^\# = e^\# + x^\# + x^{\#\#} + e \times x + e \times x^\# + x \times x^\# = 0$ since $e^\# = 0 = x^{\#\#}$, $e \times x = 0$ by (28), $e \times x^\# = T(x^\#)f - x^\# = -x^\#$ since $T(\mathscr{J}_0) = 0$, and $x \times x^\# = [T(x^\#)T(x) - N(x)]1 - T(x^\#)x - T(x)x^\# = 0$ by (26) and (32). So e' is a primitive idempotent and $T(\mathscr{J}_0(e')) = T(\mathscr{J} U_{f'}) = T(\mathscr{J}, f') = T(\mathscr{J}, f - x - x^\#) = -T(\mathscr{J}, x^\#)$ since $T(x, \mathscr{J}) = T(x, \mathscr{J}_{1/2}) = T(x \times \mathscr{J}_{1/2}) \subset T(\mathscr{J}_0) = 0$. Since $x^\#$ is not an absolute zero divisor, $x^{\#\#} = 0$ implies $T(\mathscr{J}, x^\#) \neq 0$. Accordingly we may assume the existence of a primitive idempotent e_1 such that $\mathscr{J}_0(e_1)$ contains a primitive idempotent e_2. Two primitive idempotents e_1 and e_2 are said to be *orthogonal* ($e_1 \perp e_2$) if $e_2 \in \mathscr{J}_0(e_1)$. The definition is symmetric, that is $e_2 \in \mathscr{J}_0(e_1)$ implies $e_1 \in \mathscr{J}_0(e_2)$. Indeed $e_2 = T(e_2, 1-e_1)(1-e_1) - e_1 \times e_2$ by (27) $= T(e_2)(1-e_1) - e_1 \times e_2 = 1 - e_1 - e_1 \times e_2$ and so $e_1 = 1 - e_2 - e_1 \times e_2 = T(e_1, 1-e_2)(1-e_2) - e_2 \times e_1 = e_1 U_{1-e_2} \in \mathscr{J}_0(e_2)$. Let $e_3 = 1 - e_1 - e_2 = e_1 \times e_2$; e_3 is a primitive idempotent by (9) and (10). We have $e_1 \perp e_2$, $e_1 \perp e_3$ and $e_2 \perp e_3$ since the definition is symmetric. By (21), $e_1 \times e_3 = e_2$ and $e_2 \times e_3 = e_1$. Assume from now on that $i, j, k \in \{1, 2, 3\}$ and are distinct. Define $\mathscr{J}_{ii} = \mathscr{J}_1(e_i)$, $\mathscr{J}_{ij} = \mathscr{J} U_{e_i, e_j}$. Now $x U_{e_i, e_j} U_{e_i, e_k} = [T(x, e_i)e_j + T(x, e_j)e_i - e_k \times x] U_{e_i, e_k}$ $= T(x, e_i)[T(e_j, e_i)e_k + T(e_j, e_k)e_i - e_j \times e_j] + T(x, e_j)[T(e_i, e_i)e_k + T(e_i, e_k)e_i - e_j \times e_i] -$ $T(e_k \times x, e_i)e_k - T(e_k \times x, e_k)e_i + (e_k \times e_i) \times (e_k \times x) = -T(e_k \times x, e_i)e_k - e_k^\# \times (e_i \times x) +$ $T(e_k^\#, e_i)x + T(e_k^\#, x)e_i + T(e_i \times x, e_k)e_k = 0$ by (11), (12) and (25), so U_{e_i, e_j} and U_{e_i, e_k} are orthogonal operators. Moreover $x U_{e_i, e_j}^2 = [T(x, e_i)e_j + T(x, e_j)e_i - e_k \times x] U_{e_i, e_j}$ $= T(x, e_i)[T(e_j, e_i)e_j + T(e_j, e_j)e_i - e_k \times e_j] + T(x, e_j)[T(e_i, e_i)e_j + T(e_i, e_j)e_i - e_k \times e_i] -$ $T(x \times e_k, e_i)e_j - T(x \times e_k, e_j)e_i + e_k \times (e_k \times x) = -T(x, e_j)e_j - T(x, e_i)e_i + [T(e_k^\# \times x) +$ $T(e_k)T(e_k \times x)]1 - T(x)e_k^\# - e_k^\# \times x - T(e_k^\#)x - T(e_k \times x)e_k - T(e_k)e_k \times x = -T(x, e_j)e_j -$ $T(x, e_i)e_i + T(x, e_k \times 1)1 - T(x, e_k \times 1)e_k - e_k \times x = T(x, e_i)e_j + T(x, e_j)e_i - e_k \times x = x U_{e_i, e_j}$ by (5), (12), (21) and (25), and we have $U_{e_i, e_j}^2 = U_{e_i, e_j}$. Also $U_{e_i} \perp U_{e_i, e_j + e_k} = U_{e_i, e_j} + U_{e_i, e_k}$, so $0 = U_{e_i}(U_{e_i, e_j} + U_{e_i, e_k})U_{e_i, e_j} = U_{e_i} U_{e_i, e_j}$ and similarly $U_{e_i, e_j} U_{e_i} = 0$. Hence $U_{e_i} \perp U_{e_i, e_j}$. Then $\mathscr{J} = \mathscr{J} U_{e_i} \oplus \mathscr{J} U_{e_i, e_j + e_k} \oplus \mathscr{J} U_{e_j + e_k} = \mathscr{J} U_{e_i} \oplus (\mathscr{J} U_{e_i, e_j} \oplus \mathscr{J} U_{e_i, e_k})$ $\oplus (\mathscr{J} U_{e_j} \oplus \mathscr{J} U_{e_j, e_k} \oplus \mathscr{J} U_{e_k})$. Combining this with the previous results on the Peirce decomposition of $\mathscr{J}$ with respect to a primitive idempotent e we obtain

$$\mathscr{J} = \bigoplus \sum_{i=1}^{3} \mathscr{J}_{ii} \oplus \sum_{i<j} \mathscr{J}_{ij}, \quad \mathscr{J}_{ij} = \mathscr{J}_{1/2}(e_i) \cap \mathscr{J}_{1/2}(e_j), \quad \mathscr{J}_{ii} = \Phi e_i.$$

$$T(\mathscr{J}_{ij}, \mathscr{J}_{ik}) = 0.$$

$$(33) \qquad e_i \times e_j = e_k.$$

$$e_i \times x = -x, \quad e_j \times x = 0 = e_k \times x \quad \text{for } x \in \mathscr{J}_{jk}.$$

$$x^\# = S(x)e_i \quad \text{for } x \in \mathscr{J}_{jk}.$$

Let $a \in \mathscr{J}_{ij}$, $b \in \mathscr{J}_{jk}$, then $a, b \in \mathscr{J}_{1/2}(e_j)$, so $a \times b \in \mathscr{J}_0(e_j)$. But $T(a \times b, e_i) = T(b, a \times e_i)$ $= T(b, 0) = 0$ and $T(a \times b, e_k) = T(a, b \times e_k) = T(a, 0) = 0$ by (33) and we have

(34) $a \times b \in \mathscr{J}_{ik}$ for $a \in \mathscr{J}_{ij}$, $b \in \mathscr{J}_{jk}$.

If $\mathscr{J}_{ij} = \{0\} = \mathscr{J}_{jk}$ then $\mathscr{J}_{1/2}(e_j) = 0$ and we have case (ii). Assume that only one, say, $\mathscr{J}_{ik} = \{0\}$, and $\mathscr{J}_{ij} \neq \{0\}$, $\mathscr{J}_{jk} \neq \{0\}$. Arguing as above we may pick $a \in \mathscr{J}_{1/2}(e_i)$ $= \mathscr{J}_{ij}$, $b \in \mathscr{J}_{1/2}(e_k) = \mathscr{J}_{jk}$ with $a^\# = S(a)e_k \neq 0$, $b^\# = S(b)e_i \neq 0$; $a \times b \in \mathscr{J}_{ik}$ and so $a \times b = 0$. But $(a+b)^\# = a^\# + a \times b + b^\# = S(a)e_k + S(b)e_i$ and $(a+b)^{\#\#} = S(a)S(b)e_i \times e_k$ $= S(a)S(b)e_j \neq 0$ contradicting (32). Therefore if $\mathscr{J}$ is not of the form (i) or (ii), $\mathscr{J}_{12}$, $\mathscr{J}_{23}$, $\mathscr{J}_{31}$ are nonzero and by the argument on p. 97 each contains an element whose $\#$ is nonzero. Let $a \in \mathscr{J}_{ij}$, $b \in \mathscr{J}_{ik}$. By (10) $(a \times b)^\# = T(a^\#, b)b + T(b^\#, a)a$ $- a^\# \times b^\# = -S(a)S(b)e_k \times e_j = -S(a)S(b)e_i$, so $S(a \times b) = T((a \times b)^\#) = -S(a)S(b)$ and we have

(35) $S(a \times b) = -S(a)S(b)$ for $a \in \mathscr{J}_{ij}$, $b \in \mathscr{J}_{jk}$.

By (21), $a\times(a\times b)=[T(a^{\#}\times b)+T(a)T(a\times b)]1-T(b)a^{\#}-a^{\#}\times b-T(a^{\#})b-T(a\times b)a-T(a)a\times b=T(S(a)e_k\times b)1-S(a)e_k\times b-S(a)b=-S(a)b$ by (33) and (34), so we have

(36) $a\times(a\times b)=-S(a)b$ for $a\in\mathscr{J}_{ij}$, $b\in\mathscr{J}_{jk}$.

Bilinearize (36) to get

(37) $a\times(b\times c)+b\times(a\times c)=-S(a,b)c$ for $a,b\in\mathscr{J}_{ij}$, $c\in\mathscr{J}_{jk}$.

Pick $u\in\mathscr{J}_{12}$, $v\in\mathscr{J}_{31}$ with $S(u)\neq 0$, $S(v)\neq 0$ and let $\gamma_{12}=-S(u)$, $\gamma_{31}=-S(v)$ and $\gamma_{23}=(\gamma_{31}\gamma_{12})^{-1}$. Then $u\times v\in\mathscr{J}_{23}$, $S(u\times v)=-S(u)S(v)=-\gamma_{12}\gamma_{31}=-\gamma_{23}^{-1}$. Let $w=\gamma_{23}u\times v$, so $S(w)=-\gamma_{23}$. Denote u, v, w by $1_{[12]}, 1_{[31]}, 1_{[23]}$ respectively. By (33) and (36) we have

(38) $1_{[12]}^{\#}=-\gamma_{12}e_3$, $1_{[23]}^{\#}=-\gamma_{23}e_1$, $1_{[31]}^{\#}=-\gamma_{31}e_2$.

(39) $\gamma_{12}\gamma_{23}\gamma_{31}=1$.

$$\begin{aligned}1_{[31]}\times 1_{[12]} &= \gamma_{31}\gamma_{12}1_{[23]},\\ 1_{[12]}\times 1_{[23]} &= \gamma_{12}\gamma_{23}1_{[31]},\\ 1_{[23]}\times 1_{[31]} &= \gamma_{23}\gamma_{31}1_{[12]}.\end{aligned}\tag{40}$$

Let $\mathscr{C}=\mathscr{J}_{23}$; $\mathscr{C}$ is a Φ vector space with a distinguished element $1=1_{[23]}$. Define a multiplication in $\mathscr{C}$ by

(41) $ab=(1_{[31]}\times a)\times(1_{[12]}\times b)$ for $a,b\in\mathscr{C}=\mathscr{J}_{23}$.

By (40) and (36), $1b=(1_{[31]}\times 1_{[23]})\times(1_{[12]}\times b)=\gamma_{23}\gamma_{31}1_{[12]}\times(1_{[12]}\times b)=\gamma_{23}\gamma_{31}\gamma_{12}b=b$. Similarly $a1=a$. Bilinearity of the product follows from the definition of $\times$. Define a norm on $\mathscr{C}$ by

(42) $n(a)=-\gamma_{23}^{-1}S(a)$, $a\in\mathscr{C}=\mathscr{J}_{23}$.

Then $n(1)=1$ and $n(ab)=-\gamma_{23}^{-1}S((1_{[31]}\times a)\times(1_{[12]}\times b))=\gamma_{23}^{-1}S(1_{[31]}\times a)S(1_{[12]}\times b)=\gamma_{23}^{-1}\gamma_{12}\gamma_{31}S(a)S(b)=\gamma_{23}^{-2}S(a)S(b)=n(a)n(b)$ by (35). As S is a nondegenerate quadratic form on $\mathscr{J}_{23}$, so is n on $\mathscr{C}$ and $\mathscr{C}$ is a composition algebra. It has an involution $\bar{a}=t(a)1-a=n(a,1)1-a=-\gamma_{23}^{-1}S(a,1)1_{[23]}-a=-\gamma_{23}^{-1}T(a\times 1_{[23]})1_{[23]}-a$.

(43) $\bar{a}=-\gamma_{23}^{-1}T(a\times 1_{[23]})1_{[23]}-a$, $a\in\mathscr{C}=\mathscr{J}_{23}$.

Define bijective mappings from $\mathscr{C}$ to $\mathscr{J}_{12}$ and $\mathscr{J}_{31}$:

$$\begin{aligned}\varphi:\ \mathscr{J}_{23}\to\mathscr{J}_{31}\text{ by}\quad (a)\varphi &= \gamma_{31}1_{[12]}\times\bar{a},\\ \varphi^{-1}:\ \mathscr{J}_{31}\to\mathscr{J}_{23}\text{ by}\quad (b)\varphi^{-1} &= \gamma_{23}\overline{1_{[12]}\times b},\\ \psi:\ \mathscr{J}_{23}\to\mathscr{J}_{12}\text{ by}\quad (a)\psi &= \gamma_{12}1_{[31]}\times\bar{a},\\ \psi^{-1}:\ \mathscr{J}_{12}\to\mathscr{J}_{23}\text{ by}\quad (c)\psi^{-1} &= \gamma_{23}\overline{1_{[31]}\times c}\end{aligned}$$

where $a\in\mathscr{J}_{23}$, $b\in\mathscr{J}_{31}$, $c\in\mathscr{J}_{12}$. Then

$$a\xrightarrow{\varphi}\gamma_{31}1_{[12]}\times\bar{a}\xrightarrow{\varphi^{-1}}\gamma_{23}\gamma_{31}\overline{1_{[12]}\times(1_{[12]}\times\bar{a})}=\gamma_{23}\gamma_{31}\gamma_{12}\bar{\bar{a}}=a$$

and

$$b\xrightarrow{\varphi^{-1}}\gamma_{23}\overline{1_{[12]}\times b}\xrightarrow{\varphi}\gamma_{23}\gamma_{31}1_{[12]}\times(1_{[12]}\times b)=\gamma_{23}\gamma_{31}\gamma_{12}b=b$$

by (36) and (39). Similarly $(a\psi)\psi^{-1}=a$ and $(c\psi^{-1})\psi=c$. Note that the $1_{[ij]}$'s are mapped onto one another. Denote $a\in\mathscr{C}=\mathscr{J}_{23}$ by $a_{[23]}$, $a\varphi$ by $a_{[31]}$ and $a\psi$ by $a_{[12]}$. Let

$$x = \sum_{i=1}^{3} \alpha_i e_i + \sum_{(123)} a_{i[jk]}$$

where $\sum_{(123)}$ denotes the sum over cyclic permutations of (123). To compute $x^{\#}$ it suffices to consider the last three terms of x since the others are already known from (33). By (33), $a^{\#}_{1[23]}=S(a_1)e_1=-\gamma_{23}n(a_1)e_1$. By (10) and (33), $(1_{[12]}\times\bar{a}_2)^{\#}$ $=T(1^{\#}_{[12]}, \bar{a}_2)\bar{a}_2+T(\bar{a}_2^{\#}, 1_{[12]})1_{[12]}-S(\bar{a}_2)e_1\times(-\gamma_{12}e_3)=\gamma_{12}S(\bar{a}_2)e_2$, hence

$$a^{\#}_{2[31]} = \gamma_{31}^2(1_{[12]}\times\bar{a}_2)^{\#} = \gamma_{31}\gamma_{23}^{-1}S(\bar{a}_2)e_2 = -\gamma_{31}n(\bar{a}_2)e_2 = -\gamma_{31}n(a_2)e_2.$$

Similarly $a^{\#}_{3[12]}=\gamma_{12}^2(1_{[31]}\times\bar{a}_3)^{\#}=-\gamma_{12}n(a_3)e_3$. By (41),

$$a_{3[12]} \times a_{2[31]} = \gamma_{12}\gamma_{31}(1_{[31]}\times\bar{a}_3)\times(1_{[12]}\times\bar{a}_2) = \gamma_{23}^{-1}\overline{(a_2a_3)}_{[23]}.$$

Also $a_{2[31]}\times a_{1[23]}=\gamma_{31}(\bar{a}_2\times 1_{[12]})\times a_1=\gamma_{12}^{-1}\overline{(a_1a_2)}_{[12]}$ since $\gamma_{12}^{-1}\overline{(a_1a_2)}_{[12]}=a_1a_2\times 1_{[31]}$ $=((a_1\times 1_{[31]})\times(a_2\times 1_{[12]}))\times 1_{[31]}$ by (41) $=-((a_1\times 1_{[31]})\times 1_{[31]})\times(a_2\times 1_{[12]})-$ $S(a_2\times 1_{[12]}, 1_{[31]})(a_1\times 1_{[31]})$ by (37) $=-\gamma_{31}a_1\times(a_2\times 1_{[12]})-S(a_2, \gamma_{23}^{-1}1_{[23]})(a_1\times 1_{[31]})$ by (36), (12) and (13) $=\gamma_{31}a_1\times(\bar{a}_2\times 1_{[12]})$ by (43). Similarly

$$a_{1[23]}\times a_{3[12]} = \gamma_{12}a_1\times(1_{[31]}\times\bar{a}_3) - \gamma_{31}^{-1}\overline{(a_3a_1)}_{[31]}.$$

So

(44)
$$\begin{aligned} x^{\#} = {} & (\alpha_2\alpha_3-\gamma_{23}n(a_1))e_1+(\alpha_3\alpha_1-\gamma_{31}n(a_2))e_2+(\alpha_1\alpha_2-\gamma_{12}n(a_3))e_3 \\ & +(\gamma_{23}^{-1}\overline{a_2a_3}-\alpha_1a_1)_{[23]}+(\gamma_{31}^{-1}\overline{a_3a_1}-\alpha_2a_2)_{[31]}+(\gamma_{12}^{-1}\overline{a_1a_2}-\alpha_3a_3)_{[12]}. \end{aligned}$$

Let γ_1 be an arbitrary nonzero element of Φ, $\gamma_2=\gamma_1\gamma_{12}^{-1}$, $\gamma_3=\gamma_1\gamma_{31}$, so $\gamma_{12}=\gamma_2^{-1}\gamma_1$, $\gamma_{31}=\gamma_1^{-1}\gamma_3$ and $\gamma_{23}=\gamma_3^{-1}\gamma_2$. Then if

$$X = \begin{pmatrix} \alpha_1 & a_3 & \gamma_1^{-1}\gamma_3\bar{a}_2 \\ \gamma_2^{-1}\gamma_1\bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \gamma_3^{-1}\gamma_2\bar{a}_1 & \alpha_3 \end{pmatrix} \in \mathscr{H}(\mathscr{C}_3, \mathscr{J}_\gamma),$$

X has the same $\#$ as the above element $x\in\mathscr{J}(N, \#, 1)$. The two algebras have the same underlying spaces and trace form T (by (33)), one needs only T on the Peirce spaces and $T(a_{[ij]}, b_{[ij]})=T(a_{[ij]})T(b_{[ij]})-T(a_{[ij]}\times b_{[ij]})=\gamma_{ij}t(a, b)$ (by (44)) and therefore the same U. Thus they are isomorphic and this completes the proof of Theorem 1. (Note that (32) and (3) imply that the norms are the same.)

3. **Springer's Theorem.** In [2] Faulkner proves the following theorem.

THEOREM 2 (ALBERT-JACOBSON). *Let $\mathscr{J}$ be a reduced central simple exceptional quadratic Jordan algebra. If $\mathscr{J}\cong\mathscr{H}(\mathscr{C}_3, J_\gamma)$ and $\mathscr{J}\cong\mathscr{H}(\mathscr{C}_3', J_{\gamma'})$ then $\mathscr{C}\cong\mathscr{C}'$.*

We wish to give next a simple proof of Springer's Theorem.

THEOREM 3 (SPRINGER [9]). *Let $\mathscr{J}$ and $\mathscr{J}'$ be reduced central simple exceptional quadratic Jordan algebras, $\mathscr{J} \cong \mathscr{H}(\mathscr{C}_3, J_\gamma)$, $\mathscr{J}' \cong \mathscr{H}(\mathscr{C}'_3, J_{\gamma'})$ then $\mathscr{J} \cong \mathscr{J}'$ if and only if $\mathscr{C} \cong \mathscr{C}'$ and $\mathscr{J}$ and $\mathscr{J}'$ have equivalent quadratic forms S.*

Springer [9] uses $Q(x)=\frac{1}{2}T(x^2)$ rather than $S(x)$, however, apart from considerations of characteristic, $S(x)$ seems more natural in view of Lemma 2 and of [6]. His proof makes use of spin groups and can also be found in Chapter 9 of [3]. McCrimmon [7] has given shorter proofs of Theorems 2 and 3 when the characteristic of $\Phi \neq 2$.

Necessity follows from the Albert-Jacobson Theorem. We have $\mathscr{J} \cong \mathscr{H}(\mathscr{C}_3, J_\gamma)$, $\mathscr{J}' \cong \mathscr{H}(\mathscr{C}_3, J_{\gamma'})$. If $\mathscr{C}$ is split then $n(\mathscr{C})=\Phi$ and it follows from the proof of Theorem 1 that γ_{12} and γ_{31} can be picked $=1$, so that $\gamma_{23}=1$ and $\mathscr{J} \cong \mathscr{H}(\mathscr{C}_3)$, similarly for $\mathscr{J}'$. So $\mathscr{C}$ split implies that all algebras $\mathscr{H}(\mathscr{C}_3, J_\gamma)$ are isomorphic (to $\mathscr{H}(\mathscr{C}_3)$). We may therefore assume that $\mathscr{C}$ is a division octonion algebra. We will need the Witt-Arf Theorem, e.g. [9, p. 12]. Let Q be a nondegenerate quadratic form on a Φ vector space V, $R=\{x \in V \mid Q(x, V)=0\}$ the radical of Q. An isometry s of a subspace W into V is called *admissible* if s can be extended to an isometry s' of $W+R$ in V such that s' is the identity on R.

THEOREM (WITT-ARF). *Any admissible isometry of a subspace W into V can be extended to an orthogonal transformation of V.*

Let $\mathscr{J}=\mathscr{H}(\mathscr{C}_3, J_\gamma)$. Let $x=\sum_{i=1}^3 \alpha_i e_i+\sum_{(123)} a_{i[jk]}$, $y=\sum_{i=1}^3 \beta_i e_i+\sum_{(123)} b_{i[jk]}$, it follows easily from (44) that $S(x, y)=\sum_{(123)} [(\alpha_i+\alpha_j)\beta_k-\gamma_j^{-1}\gamma_i t(a_k, b_k)]$. If $S(x, \mathscr{J}) =0$, the nondegeneracy of t implies $a_1=a_2=a_3=0$ and $\alpha_1+\alpha_2=\alpha_2+\alpha_3=\alpha_3+\alpha_1=0$. Hence the radical of $S=\{0\}$ if the characteristic of $\Phi \neq 2$, $\Phi 1$ if the characteristic of $\Phi=2$. We wish to show that any primitive idempotent e of $\mathscr{J}$ can be embedded in a system of mutually orthogonal idempotents. By the argument on p. 97, we may assume that the characteristic of Φ is 2 and we must show that $T(\mathscr{J}_0)\neq 0$. Since f does not belong to the radical of S, $S(x,f)\neq 0$ for some $x \in \mathscr{J}$. But $S(x,f) =T(x\times f)=T(x)T(f)-T(x,f)=2T(x)-T(x,f)=T(x,f)$. By (25) we may assume $x \in \mathscr{J}_0$. Therefore $T(x)=T(x,f)\neq 0$ and $T(\mathscr{J}_0)\neq 0$.

Let e_1 be a primitive idempotent of $\mathscr{J}$. Pick $e_2 \perp e_3$ primitive idempotents of $\mathscr{J}_0(e_1)$; $\mathscr{J}_0(e_1)=\Phi e_2 \oplus \Phi e_3 \oplus \mathscr{J}_{23}$, $S(\alpha_2 e_2+\alpha_3 e_3+a_{1[23]})=\alpha_2\alpha_3-\gamma_3^{-1}\gamma_2 n(a_1)$. Define the *norm class* of e_1 to be $\kappa(e_1)=-\gamma_3^{-1}\gamma_2 n(\dot{\mathscr{C}})$, where $\dot{\mathscr{C}}=\mathscr{C}-\{0\}$. By the Witt-Arf Theorem, $\kappa(e_1)$ depends only on the restriction of S to $\mathscr{J}_0(e_1)$.

LEMMA 3. *Let $\mathscr{J} \cong \mathscr{H}(\mathscr{C}_3, J_\gamma)$, $\mathscr{J}' \cong \mathscr{H}(\mathscr{C}_3, J_{\gamma'})$ with equivalent forms S, then any two primitive idempotents $e \in \mathscr{J}$ and $e' \in \mathscr{J}'$ with $\kappa(e)=\kappa(e')$ may be mapped into each other by an isomorphism of $\mathscr{J}'$ onto $\mathscr{J}$.*

Proof. We may assume that $e=e_1$, $e'=e'_1$ where e_1, e_2, e_3 and e'_1, e'_2, e'_3 are the idempotents in the above coordinatizations of $\mathscr{J}$ and $\mathscr{J}'$. Since $\kappa(e)=\kappa(e')$ there is an isometry between $\mathscr{J}'_{23}$ and $\mathscr{J}_{23}$ which can then be extended to an isometry of $\Phi e'_1 \oplus \Phi e'_2 \oplus \Phi e'_3 \oplus \mathscr{J}'_{23} = W'$ to $\Phi e_1 \oplus \Phi e_2 \oplus \Phi e_3 \oplus \mathscr{J}_{23} = W$. So the Witt-Arf Theorem implies that the above isometry can be extended to yield an isometry of $W'^{\perp} = \mathscr{J}'_{12} + \mathscr{J}'_{31}$ to $W^{\perp} = \mathscr{J}_{12} \oplus \mathscr{J}_{31}$. Thus there exists an $x \in W^{\perp}$ with $S(x) = -\gamma_2'^{-1}\gamma_1' = S(1'_{[12]})$, $1'_{[12]} \in J'_{12}$; $S(x) = T(x^{\#})$, so $x^{\#} \neq 0$, $x^{\#} \in \mathscr{J}_0(e_1)$, $x^{\#\#} = 0$ therefore $g_3 = S(x)^{-1}x^{\#}$ is a primitive idempotent in $\mathscr{J}_0(e_1)$. Let $g_1 = e_1$, $g_2 = 1 - g_1 - g_3$; by (28) and (5) $x \in \mathscr{J}_{1/2}(g_1)$ implies $(g_2 + g_3) \times x = -x$. But by (14), (26) and (32), $g_3 \times x = S(x)^{-1}x^{\#} \times x = S(x)^{-1}([T(x^{\#})T(x) - N(x)]1 - T(x^{\#})x - T(x)x^{\#}) = -S(x)^{-1}T(x^{\#})x = -x$. Therefore $g_2 \times x = 0$ and since $T(x) = 0$, (28) implies $x \in \mathscr{J}_{1/2}(g_2)$; $x \in \mathscr{J}_{1/2}(g_1) \cap \mathscr{J}_{1/2}(g_2) = \mathscr{J}_{12}$ (re the g_i's), $S(x) = -\gamma_2'^{-1}\gamma_1'$. We still have $\kappa(g_1) = \kappa(e_1) = \kappa(e'_1)$, so there is a $y \in \mathscr{J}_{23}$ with $S(y) = -\gamma_3'^{-1}\gamma_2'$. This yields a coordinatization of $\mathscr{J}$ re $e_1 = g_1, g_2, g_3$ with the same γ_i''s as the coordinatization of $\mathscr{J}'$.

COROLLARY. *Let $\mathscr{J}$ be as above, then two primitive idempotents e and e′ are in the same orbit under the automorphism group of $\mathscr{J}$ if and only if $\kappa(e) = \kappa(e')$.*

The following lemma will complete the proof of Theorem 3.

LEMMA 4. *If $\mathscr{J} \cong \mathscr{H}(\mathscr{C}_3, J_\gamma)$, $\mathscr{J}' \cong \mathscr{H}(\mathscr{C}_3, J_{\gamma'})$ have equivalent forms S then there exist primitive idempotents $e \in \mathscr{J}$, $e' \in \mathscr{J}'$ with $\kappa(e) = \kappa(e')$.*

Proof. Assume $e' = e'_1$, then $\kappa(e') = -\gamma_3'^{-1}\gamma_2' n(\dot{\mathscr{C}})$. The Witt-Arf Theorem allows us to extend the isometry between $\Phi e'_1 \oplus \Phi e'_2 \oplus \Phi e'_3$ and $\Phi e_1 \oplus \Phi e_2 \oplus \Phi e_3$ by an isometry between $\mathscr{J}'_{12} \oplus \mathscr{J}'_{23} \oplus \mathscr{J}'_{31}$ and $\mathscr{J}_{12} \oplus \mathscr{J}_{23} \oplus \mathscr{J}_{31}$. So there is an $x \in \mathscr{J}_{12} \oplus \mathscr{J}_{23} \oplus \mathscr{J}_{31}$ with $S(x) = -\gamma_3^{-1}\gamma_2$. If $x \in \mathscr{J}_{1/2}(e_i)$ for some $i = 1$, 2 or 3 then argue as in the proof of Lemma 3 to get a primitive idempotent e with $\kappa(e) = S(x)n(\dot{\mathscr{C}})$. If not then $x = a_{1[23]} + a_{2[31]} + a_{3[12]}$ with $n(a_1)n(a_2)n(a_3) \neq 0$,

$$S(x) = -[\gamma_3^{-1}\gamma_2 n(a_1) + \gamma_1^{-1}\gamma_3 n(a_2) + \gamma_2^{-1}\gamma_1 n(a_3)],$$

$N(x) = t(a_1a_2a_3)$. We proceed to show that we may assume $N(x) = 0$. If this is not the case, recoordinatizing if necessary, we may assume $a_1 = a_2 = 1$. Pick $b \in \dot{\mathscr{C}}$ with $t(ba_3) = t(\bar{b}, a_3) = 0$. Consider $y = b_{[23]} + \bar{b}_{[31]} + ba_{3[12]}$, $S(y) = n(b)S(x)$ and $N(y) = t(b\bar{b}(ba_3)) = n(b)t(ba_3) = 0$. Thus we may assume that we have an $x \in \mathscr{J}_{12} \oplus \mathscr{J}_{23} \oplus \mathscr{J}_{31}$ with $S(x) \in \kappa(e')$, $N(x) = 0$; so $x^{\#\#} = 0$ and $g = S(x)^{-1}x^{\#}$ is a primitive idempotent. By (14) $g \times x = -x$ and since $T(x) = 0$, (29) implies $x \in \mathscr{J}_0(g)$. If $T(e_1, g) = 1$ then the coefficient of e_1 in $x^{\#}$ is $S(x)$ and letting $y = a_{1[23]}$, $y^{\#} = S(y)e_1 = S(x)e_1$ and $\kappa(e') = \kappa(e_1)$. If not, consider e_1U_f where $f = 1 - g$; $e_1U_f \in \mathscr{J}_0(g)$, $(e_1U_f)^{\#} = e_1^{\#}U_{f^{\#}} = 0$, $T(e_1U_f) = T(e_1U_f, 1) = T(e_1, f) = T(e_1, 1 - g) = 1 - T(e_1, g) \neq 0$. Therefore $g_2 = T(e_1U_f)^{-1}e_1U_f$ is a primitive idempotent in $\mathscr{J}_0(g)$, $T(e_1U_f, x) = T(e_1, xU_f) = T(e_1, x) = 0$. Now $x \in \mathscr{J}_0(g)$, $T(x) = 0$ and $T(g_2, x) = 0$ imply $x \in (\Phi g_2 + \Phi g_3)^{\perp} \cap \mathscr{J}_0(g)$ where $g_3 = f - g_2$ and we have $\kappa(g) = \kappa(e')$. This completes the proof of the lemma.

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